JOURNAL OF APPROXIMATION THEORY 13, 84–104 (1975)

# Notes on Spline Functions V. Orthogonal or Legendre Splines*

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DEDICATED TO PROFESSOR G. G. LORENTZ ON THE OCCASION OF HIS SIXTY-FIFTH BIRTHDAY

We follow the general program of generalizing to spline functions problems concerning polynomials. Let $N$ and $n$ be natural numbers, and let $q = [(n-1)/2]$, the integral part of $(n-1)/2$. The orthogonalization of the powers $x^k$ $(k = 0, 1,..., N + 2q)$ in the interval $[0, 1]$ by the Gram–Schmidt process leads to the monic Legendre polynomials $X_k(x)$ $(k = 0,..., N + 2q)$. We now consider the class $\mathscr{S}_{n-1,N}[0, 1] = \{s(x)\}$ of splines $s(x)$ of degree $n - 1$ in $[0, 1]$. These have the $N - 1$ knots $1/N, 2/N,..., (N-1)/N$ if $n$ is even, and the $N$ knots $1/2N, 3/2N,..., (2N-1)/N$ if $n$ is odd. This family is found to depend on $N + 2q + 1$ parameters. It is shown how to construct an orthogonal basis $g_{k,N}(x)$ $(k = 0,..., N + 2q)$ for this class of splines having the following two properties: 1. $g_{k,N}(x) = X_k(x)$ if $k = 0, 1,..., n - 1$. 2. $g_{k,N}(x) = X_k(x) + 0(N^{-n})$ as $N \to \infty$, for all integer $k$. They are called the Legendre splines.

## 1. INTRODUCTION

Let $n$ and $N$ be natural numbers and let

$$\mathscr{S}_{n-1,N}[0, 1] = \{s(x)\} \tag{1.1}$$

denote the class of splines $s(x)$ of degree $n - 1$, defined in the interval $[0, 1]$ and having the knots

$$1/N, 2/N,..., (N-1)/N \text{ if } n \text{ is even} \tag{1.2}$$

and

$$1/2N, 3/2N,..., (2N-1)/2N \text{ if } n \text{ is odd.} \tag{1.3}$$

This is a linear family depending on $n + N - 1$ parameters if $n$ is even, and on $n + N$ parameters if $n$ is odd. Writing

$$q = [(n-1)/2], \tag{1.4}$$

* Sponsored by the United States Army under Contract No. DA-31-124-ARO-D-462.

we find that in either case

$$\textit{the class } (1.1) \textit{ depends on } N + 2q + 1 \textit{ parameters.} \tag{1.5}$$

The purpose of the present note is to construct an orthogonal basis for the class (1.1). To obtain it we could start from any sequence of $N + 2q + 1$ linearly independent elements of (1.1) and apply to it the Gram–Schmidt orthogonalization process (see [1, Sections 3.13 and 5.14]). A worthwhile problem arises, however, if we place certain additional requirements on the basis so obtained. Let

$$X_k(x), \qquad (k = 0, 1, \ldots), \tag{1.6}$$

be the monic Legendre polynomials for the interval [0, 1], hence $X_0 = 1$, $X_1 = x - (\frac{1}{2})$, $X_2 = x^2 - x + (\frac{1}{6})$, a.s.f. In 1965 the author had the idea that it should be possible to find for the class (1.1) an orthogonal basis

$$g_{k,N}(x), \qquad (k = 0, 1, \ldots, N + 2q), \tag{1.7}$$

having the following two properties.

PROPERTY 1. *For $k = 0, 1, \ldots, n - 1$, we should retain the Legendre polynomials, hence*

$$g_{k,N}(x) = X_k(x) \textit{ in } [0, 1], \qquad \textit{for} \quad k = 0, 1, \ldots, n - 1. \tag{1.8}$$

PROPERTY 2. *The orthogonal basis (1.7) should converge to the Legendre polynomials as we keep $n$ fixed and let $N \to \infty$, hence*

$$\lim_{N\to\infty} g_{k,N}(x) = X_k(x),$$

*for every integer $k$.*

In Section 5 we construct an orthogonal basis having the Properties 1 and 2. Its elements are called *Legendre splines*. A second construction (Section 6), based on complete spline interpolation, is shown to furnish an orthogonal basis, also enjoying the Properties 1 and 2, that is in general different from the Legendre splines. The main new idea of the paper (Section 3) concerns the cardinal spline interpolation of polynomials. The problem of least squares approximation by splines has received the attention that it deserves (see [2–5]). To set the stage for our extensive use of $B$-splines, we discuss in Section 2 a readily implemented approach to this problem. The last Section 8 illustrates the brave behavior of cubic splines under difficult circumstances.

## 2. The Least-Squares Approximation by Splines

In order to avoid fractional knots we consider the splines $S(x) = s(x/N)$, where $s(x)$ belongs to the class (1.1). Accordingly, let

$$\mathscr{S}_{n-1}[0, N] = \{S(x)\}, \tag{2.1}$$

be the class of splines of degree $n - 1$ in $[0, N]$, with knots $1, 2,..., N - 1$, if $n$ is even, and knots $\frac{1}{2}, \frac{3}{2},..., N - (\frac{1}{2})$, if $n$ is odd. If $f(x) \in L_2(0, N)$, we are to find the element $S(x)$ of (2.1) such as to minimize the integral

$$\int_0^N (f(x) - S(x))^2\, dx. \tag{2.2}$$

As simple a solution as any seems to be the following. It is based on the remark that every element of the class (2.1) admits a unique representation of the form

$$S(x) = \sum_{j=-q}^{N+q} c_j M_n(x - j) \quad \text{in} \quad [0, N]. \tag{2.3}$$

Here $M_n(x)$ is the central $B$-spline of my old paper [6, Section 3.13]. Substituting (2.3) into (2.2) we obtain the problem

$$\int_0^N \left\{ f(x) - \sum_{-q}^{N+q} c_j M_n(x - j) \right\}^2 dx = \text{minimum}, \tag{2.4}$$

and the normal equations are

$$\int_0^N \left\{ f(x) - \sum_j c_j M_n(x - j) \right\} M_n(x - i)\, dx = 0. \tag{2.5}$$

Writing,

$$\lambda_{ij} = \int_0^N M_n(x - i)\, M_n(x - j)\, dx \qquad (i, j = -q,..., N + q), \tag{2.6}$$

and

$$I_i = \int_0^N f(x)\, M_n(x - i)\, dx \qquad (i = -q,..., N + q), \tag{2.7}$$

the Eqs. (2.5) become

$$\sum_{j=-q}^{N+q} \lambda_{ij} c_j = I_i \qquad (i = -q,..., N + q). \tag{2.8}$$

If $f(x)$ is defined in [0, 1] and we seek the spline

$$s(x) = \sum_{-q}^{N+q} c_j M_n(Nx - j) \qquad (0 \leqslant x \leqslant 1), \tag{2.9}$$

i.e., the general element of the class (1.1), which is the least-squares approximation of $f(x)$ in [0, 1], we find for the coefficients $c_j$ of (2.9) the equations

$$\sum_{j=-q}^{N+q} \lambda_{ij} c_j = N \int_0^1 f(x)\, M_n(Nx - i)\, dx \quad (i = -q, \ldots, N + q), \tag{2.10}$$

where the $\lambda_{ij}$ have the old meaning (2.6). These equations are to be solved on a computer. The function $M_n(x)$ is known to have its support in $(-n/2, n/2)$ where it is positive [7, Section 1], and the right sides of (2.10) are seen to be "local averages" of $f(x)$. The implementation of this method requires the numerical values of the elements of the matrix

$$A_{n,N} = \| \lambda_{ij} \|. \tag{2.11}$$

From the relation

$$\int_{-\infty}^{\infty} M_n(x - i)\, M_n(x - j)\, dx = M_{2n}(i - j) \qquad \text{(See [7, Section 1])}, \tag{2.12}$$

we find that

$$\lambda_{ij} = M_{2n}(i - j), \tag{2.13}$$

as long as we have the inclusion

$$(i - n/2, i + n/2) \cap (j - n/2, j + n/2) \subset (0, N), \tag{2.14}$$

where the left side is the support of the integrand in (2.12). To simplify our discussion we shall assume $N \geqslant n - 1$. From the symmetry of $\lambda_{ij}$ we may also assume that $i \leqslant j$. It is then seen that (2.14), and therefore (2.13), holds unless,

$$\text{either } j - n/2 < 0, \qquad \text{or else } i + n/2 > N. \tag{2.15}$$

We conclude, assuming $i \leqslant j$, that (2.13) holds unless,

$$i \leqslant j < n/2, \qquad \text{or else } N - n/2 < i \leqslant j. \tag{2.16}$$

It follows that the "irregular" values of $\lambda_{ij}$ (i.e., those not given by (2.13)) correspond to the elements of the matrix (2.11) contained in the North-West and South-East principal minors of $A$ of order $2q + 1$.

For $n = 4$, hence $q = 1$, and $N = 4$ we find that

$$\Lambda_{4,4} = 1/5040 \cdot \left\| \begin{array}{ccc:c:ccc} 20 & 129 & 60 & 1 & 0 & 0 & 0 \\ 129 & 1208 & 1062 & 120 & 1 & 0 & 0 \\ 60 & 1062 & 2396 & 1191 & 120 & 1 & 0 \\ \hdashline 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 \\ \hdashline 0 & 1 & 120 & 1191 & 2396 & 1062 & 60 \\ 0 & 0 & 1 & 120 & 1062 & 1208 & 129 \\ 0 & 0 & 0 & 1 & 60 & 129 & 20 \end{array} \right\| \qquad (2.17)$$

The irregular $\lambda_{ij}$ of the two boxed third order minors ($2q + 1 = 3$) do not depend on $N$ and must be evaluated by direct computation for each value of $n$.

There is a simple check as follows. From the relation

$$\sum_{-q}^{N+q} M_n(x - i) = 1 \qquad \text{in} \quad 0 \leqslant x \leqslant N,$$

we find that

$$N = \int_0^N \left\{ \sum_i M_n(x - i) \right\} \left\{ \sum_j M_n(x - j) \right\} dx = \sum_{i,j} \lambda_{ij},$$

in view of (2.6). We mention that the perturbed Toeplitz matrix $\Lambda_{n,N}$ is positive definite and well conditioned. See Section 8 for a numerical example.

## 3. The Cardinal Spline Interpolation of Polynomials

Let

$$\mathscr{S}_{n-1} = \{S(x)\}, \qquad (3.1)$$

denote the class of cardinal splines of degree $n - 1$, with knots at the points $j + (n/2)$, hence of functions of the form

$$S(x) = \sum_{-\infty}^{\infty} c_j M_n(x - j). \qquad (3.2)$$

Our approach to the problem of orthogonal splines requires a solution of the

Problem 1 to be stated below. It is known (see [8] or [9, Lecture 4]) that if $f(x)$, from **R** to **C**, grows at most like a power of $|x|$ as $x \to \pm\infty$, then there exists a unique $S(x) \in \mathscr{S}_{n-1}$ that also grows at most like a power of $|x|$, with the property of interpolating $f(x)$ at all integer values of $x$, hence such that $S(j) = f(j)$ for all integer $j$. We also know that this unique interpolant is given by the expansion

$$S(x) = \sum_{-\infty}^{\infty} f(j)\, L_n(x - j). \tag{3.3}$$

Here $L_n(x)$ is the fundamental function of this interpolation process and is uniquely defined by the requirements: $L_n(x)$ is of power growth and such that

$$L_n(x) \in \mathscr{S}_{n-1}, \qquad L_n(j) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \neq 0. \end{cases} \tag{3.4}$$

It actually follows that $L_n(x) \to 0$ exponentially as $|x| \to \infty$.

We consider now the special case where

$$f(x) = P(x) \in \pi_k \tag{3.5}$$

and state

PROBLEM 1. *How do we recognize that* (3.2) *is the cardinal spline interpolant of a polynomial $P(x)$ of degree $k$?*

For its solution we need a few old tools that have proved to be useful in a discussion of cardinal spline interpolation (see [7, Section 2] or [9, Lecture 4, and Section 1 of Lecture 10]). The Fourier transform of the $B$-spline $M_n(x)$ is the function

$$\psi_n(u) = \left(2 \sin \frac{u}{2} \Big/ u\right)^n. \tag{3.6}$$

From it we derive the periodic function

$$\phi_n(u) = \sum_{j=-\infty}^{\infty} \psi_n(u + 2\pi j) \tag{3.7}$$

which is also identical with the cosine polynomial

$$\phi_n(u) = \sum_{|\nu| < n/2} M_n(\nu) e^{i\nu u}. \tag{3.8}$$

This cosine polynomial is always positive. If

$$1/\phi_n(u) = \sum_{\nu=-\infty}^{\infty} \omega_\nu e^{i\nu u}, \tag{3.9}$$

is the Fourier series expansion of its reciprocal, then the fundamental function of (3.4) admits the representation

$$L_n(x) = \sum_{\nu=-\infty}^{\infty} \omega_\nu M_n(x-\nu). \tag{3.10}$$

A solution of Problem 1 is provided by the following

THEOREM 1. *Let $P(x)$ be a polynomial and let*

$$S(x) = \sum_{-\infty}^{\infty} c_j M_n(x-j) \tag{3.11}$$

*be the cardinal spline interpolant of $P(x)$. If we define the rational numbers $\gamma_{2r} = \gamma_{2r}^{(n)}$ by the power series expansion*

$$1/\phi_n(u) = \sum_{r=0}^{\infty} \gamma_{2r} u^{2r}, \tag{3.12}$$

*then the coefficients $c_j$ in* (3.11) *are given by*

$$c_j = Q(j) \qquad \textit{for all integer } j, \tag{3.13}$$

*where*

$$Q(x) = \sum_{r=0}^{\infty} (-1)^r \gamma_{2r} P^{(2r)}(x). \tag{3.14}$$

*This is a finite sum because its terms vanish as soon as $2r$ exceeds the degree of $P(x)$.*

*Proof.* By (3.3) we know that the interpolant of $P(x)$ is

$$S(x) = \sum_\nu P(\nu)\, L_n(x-\nu).$$

Using (3.10) we can write

$$S(x) = \sum_\nu P(\nu) \sum_j \omega_j M_n(x-\nu-j),$$

and replacing $j$ by $j-\nu$ in the inside sum we find that

$$S(x) = \sum_{\nu,j} P(\nu)\omega_{j-\nu}\, M_n(x-j) = \sum_j c_j M_n(x-j),$$

where

$$c_j = \sum_\nu P(\nu)\omega_{j-\nu} = \sum_\nu \omega_\nu P(j-\nu). \tag{3.15}$$

Applying Taylor's theorem we obtain

$$c_j = \sum_{\nu} \omega_\nu \{P(j) - P'(j)\nu/1! + P''(j)\nu^2/2! - \cdots\}. \tag{3.16}$$

We evidently need the values of the sums $\sigma_s = \sum_\nu \omega_\nu \nu^s$. Expanding the right side of (3.9) in powers of $u$ we obtain

$$1/\phi_n(u) = \sum_{\nu} \omega_\nu \{1 + i\nu u/1! + i^2\nu^2 u^2/2! + \cdots\}.$$

Interchanging the order of summations and comparing with (3.12) we find that $\sigma_s = 0$ if $s$ is odd, and that

$$1/(2r)! \sum_{\nu=-\infty}^{\infty} \omega_\nu \nu^{2r} = (-1)^r \gamma_{2r}\,. \tag{3.17}$$

Substituting this into (3.16) we obtain (3.13), (3.14).

*Remarks.* 1. If the degree $k$ of $P(x)$ is $< n$, then the relations (3.13), (3.14) were known before. For in this case we have $S(x) = P(x)$ for all real $x$, and it was shown in [8, Theorem 5, p. 409] that

$$c_j = \sum_{2r<n-1} (-1)^r \tilde{\gamma}_{2r} P^{(2r)}(j), \tag{3.18}$$

where the new rational coefficients $\tilde{\gamma}_{2r}$ are defined by the expansion

$$1/\psi_n(u) = \sum_{r=0}^{\infty} \tilde{\gamma}_{2r} u^{2r}. \tag{3.19}$$

Let us compare the expansions (3.12) and (3.19). From (3.7) and the fact that all terms $\psi_n(u + 2\pi j)$ in (3.7), for $j \neq 0$, have a zero of order $n$ at the origin $u = 0$, we see that

$$\gamma_{2r} = \tilde{\gamma}_{2r} \qquad \text{if} \quad 2r \leqslant n - 1. \tag{3.20}$$

Therefore, if $k \leqslant n - 1$, then (3.13), (3.14) are identical with (3.18).

2. The second remark is irrelevant for our main theme of orthogonal splines, but seems too tempting to omit entirely. We raise the question: *What happens to Theorem* 1 *if $P(x)$ is an entire function rather than a polynomial*?

From (3.14) it is clear that an answer will depend on the rate of decrease of the coefficients of the expansion (3.12), and this in turn will depend on the location of the zeros of the cosine polynomial (3.8). It was shown in

[9, Theorem 2, p. 22, Theorem 4, p. 25] that the zeros of the reciprocal (or symmetric) Laurent polynomial

$$\sum_{\nu} M_n(\nu) z^{\nu}, \qquad (-n/2 < \nu < n/2),$$

are simple and negative and that $z = -1$ is not a zero. Setting $z = e^{iu}$ we conclude from (3.8) that all the zeros of $\phi_n(u)$ in the period strip $0 \leqslant \operatorname{Re} u < 2\pi$ are on the line $\operatorname{Re} u = \pi$ and that $\phi_n(\pi) \neq 0$. We conclude that the radius of convergence $R_n$ of the series (3.12) satisfies $R_n > \pi$. It follows that for appropriate positive constants $A_n$ and $\epsilon_n$ we have

$$|\gamma_{2r}| < A_n(\pi + \epsilon_n)^{-2r} \qquad \text{for all } r. \tag{3.21}$$

This implies easily the following theorem.

THEOREM 2. *If*

$$f(x) \textit{ is an entire function of exponential type } \leqslant \pi, \tag{3.22}$$

*and if we set*

$$c_j = \sum_{r=0}^{\infty} (-1)^r \gamma_{2r} f^{(2r)}(j), \tag{2.23}$$

*then*

$$S(x) = \sum_{-\infty}^{\infty} c_j M_n(x - j), \tag{3.24}$$

*is a cardinal spline such that*

$$S(j) = f(j) \qquad \textit{for all integer } j. \tag{3.25}$$

In the present discussion we have left the realm of functions of power growth. In this wider field of functions satisfying (3.22) cardinal spline interpolation is no longer unique, as we may add to any interpolant linear combinations of so-called *eigensplines* that vanish at the integers (see [9], Lecture 3, Section 5). It would be of interest to have an intrinsic characterization of the particular interpolant $S(x)$ provided by Theorem 2. We omit the proof of Theorem 2 because we have no such characterization. However, the following example seems worthwhile. We apply Theorem 2 to the function

$$f(x) = e^{iux}, \qquad (u \text{ constant}, -\pi \leqslant u \leqslant \pi),$$

which evidently satisfies (3.22).

Using (3.23) we find

$$c_j = \sum_{r=0}^{\infty} (-1)^r \gamma_{2r}(iu)^{2r} e^{iuj} = e^{iuj} \sum_{0}^{\infty} \gamma_{2r} u^{2r} = e^{iuj}/\phi_n(u),$$

in view of (3.12). Now (3.24) becomes

$$S(x) = \sum_j e^{iuj} M_n(x-j)/\phi_n(u) = \sum_j e^{iuj} M_n(x-j) \Big/ \sum_j e^{iuj} M_n(-j),$$

by (3.8). We now recognize the interpolant $S(x)$ to be identical with the exponential Euler spline of [9, Lecture 3, Sections 5 and 6].

## 4. A Solution of Problem 1 of Section 3

A solution is explicitly stated in Theorem 1 which shows that the coefficients $c_j$ of the spline (3.11) are the values (3.13) of a polynomial $Q(x)$ of degree $k$ that is explicitly given by (3.14). This necessary condition for $S(x)$ to interpolate a polynomial is also sufficient as stated in

THEOREM 3. *The cardinal spline*

$$S(x) = \sum_{-\infty}^{\infty} Q(\nu)\, M_n(x-\nu), \qquad \textit{where} \quad Q(x) \in \pi_k, \tag{4.1}$$

*interpolates at the integers a polynomial $P(x)$ of degree $k$ which is explicitly found as follows: In terms of the coefficients of the expansion*

$$\phi_n(u) = \sum_{0}^{\infty} \delta_{2r} u^{2r}, \tag{4.2}$$

*we obtain $P(x)$ by*

$$P(x) = \sum_{0}^{\infty} (-1)^r \delta_{2r} Q^{(2r)}(x). \tag{4.3}$$

*Proof.* The Lagrange expansion (3.3) has the property that $S(x) = f(x)$ for all real $x$, whenever $f(x) \in \mathscr{S}_{n-1}$. Applying this to $f(x) = M_n(x-\nu)$ we obtain the identity

$$M_n(x-\nu) = \sum_j M_n(j-\nu)\, L_n(x-j).$$

Introducing this into (4.1) we obtain

$$S(x) = \sum_{\nu,j} Q(\nu) M_n(j-\nu)\, L_n(x-j) = \sum_j p_j L_n(x-j), \tag{4.4}$$

where $p_j$ is defined by

$$p_j = \sum_\nu Q(\nu)\, M_n(j-\nu) = \sum_\nu M_n(\nu)\, Q(j-\nu)$$

$$= \sum_{\nu=-\infty}^{\infty} M_n(\nu)\, \{Q(j) - Q'(j)\,\nu/1! + Q''(j)\,\nu^2/2! - \cdots\}. \tag{4.5}$$

Expanding the right side of (3.8) in powers of $u$ we find that

$$\phi_n(u) = \sum_\nu M_n(\nu)\, \{1 + i\nu u/1! + i^2\nu^2u^2/2! + \cdots\},$$

and comparing with (4.2) we obtain

$$1/(2r)!\, \sum_{\nu=-\infty}^{\infty} M_n(\nu)\nu^{2r} = (-1)^r\, \delta_{2r}\,.$$

Introducing these values into (4.5) we obtain that

$$p_j = \sum_{r=0}^{\infty} (-1)^r\, \delta_{2r} Q^{(2r)}(j). \tag{4.6}$$

The relations (4.4) and (4.6) show that the polynomial $P(x)$, defined by (4.3), does indeed interpolate $S(x)$ at the integers.

We mention two numerical examples, for $n = 3$ and $n = 4$. From (3.8) and the diagrams of $M_3(x)$ and $M_4(x)$ in [6, p. 71] we find that

$$\phi_3(u) = \tfrac{1}{4}(3 + \cos u) = 1 - \tfrac{1}{8}u^2 + (1/96)\, u^4 - \cdots,$$

$$\phi_4(u) = \tfrac{1}{3}(2 + \cos u) = 1 - \tfrac{1}{6}u^2 + (1/72)\, u^4 - \cdots,$$

whence,

$$1/\phi_3(u) = 1 + \tfrac{1}{8}u^2 + (1/192)\, u^4 + \cdots,$$

$$1/\phi_4(u) = 1 + \tfrac{1}{6}u^2 + (1/72)\, u^4 + \cdots.$$

Applying Theorem 1 we obtain the following:

COROLLARY 1. *If* $P(x) \in \pi_5$, *then*

$$\sum_{-\infty}^{\infty} \{P(j) - \tfrac{1}{8}P''(j) + (1/192)\, P^{(4)}(j)\}\, M_3(x-j),$$

*is the quadratic spline interpolating* $P(x)$ *at the integers.*

COROLLARY 2. *If $P(x) \in \pi_5$, then*

$$\sum_{-\infty}^{\infty} \{P(j) - \tfrac{1}{6}P''(j) + (1/72)\, P^{(4)}(j)\}\, M_4(x - j)$$

*is the cubic spline interpolating $P(x)$ at the integers.*

In the opposite direction we can apply Theorem 3 and state the following:

COROLLARY 1′. *If $Q(x) \in \pi_5$, then $\sum Q(j)\, M_3(x - j)$ is a quadratic spline interpolating at the integers the quintic polynomial*

$$P(x) = Q(x) + \tfrac{1}{8}Q''(x) + (1/96)\, Q^{(4)}(x).$$

COROLLARY 2′. *If $Q(x) \in \pi_5$, then $\sum Q(j)\, M_4(x - j)$ is a cubic spline interpolating at the integers the quintic polynomial*

$$P(x) = Q(x) + \tfrac{1}{6}Q''(x) + (1/72)\, Q^{(4)}(x).$$

We conclude this section with the cardinal spline interpolation of

$$P(x) = x^n/n!$$

by elements of $\mathscr{S}_{n-1}$. By Theorem 1 we find the interpolating spline to be (3.11), where

$$c_j = \sum_{2r \leqslant n} (-1)^r\, \gamma_{2r}\, j^{n-2r}/(n - 2r)!. \tag{4.7}$$

The remainder of this interpolation is a very classical function and we find that

$$\frac{x^n}{n!} - \sum_{-\infty}^{\infty} c_j M_n(x - j) = \begin{cases} \{\bar{B}_n(x) - B_n\}/n!, & \text{if } n \text{ is even}, \\ \bar{B}_n(x + \tfrac{1}{2})/n!, & \text{if } n \text{ is odd}. \end{cases} \tag{4.8}$$

Here $\bar{B}_n(x)$ is the periodic extension of the Bernoulli polynomial $B_n(x)$, while $B_n$ is the Bernoulli number. This remark was made before (see [8, p. 412]), but now we have the explicit coefficients (4.7).

## 5. THE LEGENDRE SPLINES AND THEIR PROPERTIES

Our Theorem 1 and 3 focus our attention on the class of splines

$$\mathscr{S}_{n-1}^{(k)} = \left\{ S(x) = \sum_{-\infty}^{\infty} Q(j) M_n(x - j); \quad \text{where} \quad Q(x) \in \pi_k \right\} \tag{5.1}$$

for every integer $k$. They may be called the class of cardinal splines of *interpolation degree k*. Evidently

$$\mathscr{S}_{n-1}^{(k)} = \pi_k \ (k = 0,\ldots, n-1), \tag{5.2}$$

while

$$\mathscr{S}_{n-1}^{(k)} \ (k = n, n+1,\ldots), \tag{5.3}$$

is an ever expanding sequence of classes whose union is far from exhausting $\mathscr{S}_{n-1}$.

We now consider the restrictions to $[0, N]$ of the elements of (5.1). Their class may also be defined by

$$\mathscr{S}_{n-1}^{(k)}[0, N] = \left\{ S(x) = \sum_{-q}^{N+q} Q(j) M_n(x-j) \quad \text{in} \quad [0, N); \quad \text{where} \quad Q(x) \in \pi_k \right\}. \tag{5.4}$$

Unlike the classes (5.3), let us show that *we may restrict k to satisfy*

$$k \leqslant N + 2q. \tag{5.5}$$

For if $k < N + 2q$ then

$$S(x) = \sum_{-q}^{N+q} c_j \, M_n(x-j) \in \mathscr{S}_{n-1}^{(k)}[0, N], \tag{5.6}$$

means that the $c_j$ are interpolated by a $Q(x) \in \pi_k$, and this is a real restriction. Already when $k = N + 2q$ we see that this can always be done. Thus

$$\mathscr{S}_{n-1}^{(k)}[0, N] = \pi_k \qquad (k = 0,\ldots, n-1), \tag{5.7}$$

while with strict inclusion

$$\pi_{n-1} \subset \mathscr{S}_{n-1}^{(n)}[0, N] \subset \mathscr{S}_{n-1}^{(n+1)}[0, N] \subset \cdots \subset \mathscr{S}_{n-1}^{(N+2q)}[0, N] = \mathscr{S}_{n-1}[0, N]. \tag{5.8}$$

We now construct an orthogonal basis of $\mathscr{S}_{n-1}[0, N]$ as follows: *We start from the monomial*

$$x^k \qquad (k \leqslant N + 2q), \tag{5.9}$$

and denote by

$$S_k(x) \qquad (k \leqslant N + 2q), \tag{5.10}$$

its unique spline interpolant within the class (5.4). By Theorem 1 we know that $S_k(x)$ is of the form (5.6) where

$$c_j = j^k - \gamma_2 k(k-1) \, j^{k-2} + \cdots. \tag{5.11}$$

*To the set* (5.10) *we apply the Gram–Schmidt process to obtain the orthogonal set of splines*

$$G_k(x) \qquad (k = 0, 1,..., N + 2q). \tag{5.12}$$

*These we call the* Legendre splines *for the interval* $[0, N]$.

We can obtain the Legendre splines for $[0, 1]$ in two different ways.

(1) We may define them by

$$g_k(x) = (1/N^k)\, G_k(Nx) \qquad (0 \leqslant x \leqslant 1). \tag{5.13}$$

(2) Alternatively we consider the

$$s_k(x) = (1/N^k)\, S_k(Nx), \tag{5.14}$$

interpolating $x^k$ at $0, 1/N,..., 1$, and orthogonalize the $s_k(x)$ by the Gram–Schmidt process to obtain the same functions $g_k(x)$.

We write $g_k(x) = g_{k,N}(x)$, as they depend on $N$, and wish to show that they enjoy the Properties 1 and 2 of Section 1. The Property (1.8) evidently holds by our construction. Let us show that (1.9) holds in the following stronger form.

THEOREM 4. *The Legendre splines* (5.13) *satisfy*

$$g_k(x) = g_{k,N}(x) = X_k(x) + O(N^{-n}) \text{ in } [0, 1], \quad \text{as} \quad N \to \infty, \tag{5.15}$$

*for every integer* $k$.

*Proof.* We shall use the known fact that the spline interpolant (5.14) of $x^k$ approximates $x^k$ as shown by

$$s_k(x) = x^k + O(N^{-n}) \text{ in } [0, 1], \quad \text{as} \quad N \to \infty. \tag{5.16}$$

Evidently (5.15) holds as long as $k < n$ in view of (1.8). Let us suppose that $g_{0,N},..., g_{k-1,N}$ have already been constructed from $s_0, s_1,..., s_{k-1}$ by the Gram–Schmidt process and found to satisfy

$$g_\nu(x) = g_{\nu,N}(x) = X_\nu(x) + O(N^{-n}) \qquad (\nu = 0,..., k - 1). \tag{5.17}$$

By the Gram–Schmidt process we find that

$$g_k(x) = s_k(x) - \sum_{\nu=0}^{k-1} \frac{(s_k, g_\nu)}{(g_\nu, g_\nu)}\, g_\nu(x). \tag{5.18}$$

Using (5.16) and (5.17) we obtain

$$g_k(x) = x^k - \sum_{\nu=0}^{k-1} \frac{(x^k, X_\nu)}{(X_\nu, X_\nu)}\, X_\nu(x) + O(N^{-n}).$$

The right side, omitting the remainder, is evidently $= X_k(x)$ and (5.15) therefore holds. This completes the proof.

## 6. An Explicit Representation of Legendre Splines

If

$$S(x) = \sum_{-q}^{N+q} c_j M_n(x-j) \qquad \text{and} \qquad \tilde{S}(x) = \sum_{-q}^{N+q} \tilde{c}_j M_n(x-j), \tag{6.1}$$

then we know that in terms of the $\lambda_{ij}$, defined by (2.6), we have

$$\int_0^N S(x)\,\tilde{S}(x)\,dx = \sum_{i,j=-q}^{N+q} \lambda_{ij} c_i \tilde{c}_j . \tag{6.2}$$

We define in the space $\mathbf{R}^{N+2q+1}$ of vectors

$$\underline{c} = (c_j) \qquad (j = -q,\ldots, N+q), \tag{6.3}$$

the inner product

$$(\underline{c}, \tilde{\underline{c}}) = \sum \lambda_{ij} c_i \tilde{c}_j , \tag{6.4}$$

and with respect to this inner product we are to orthogonalize the vectors

$$(j^k), \qquad (k = 0, 1,\ldots, N+2q). \tag{6.5}$$

This can be done by the following well-known general procedure.

Theorem 5. *We denote by $(r, \nu)$ the inner product*

$$(r, \nu) = ((j^r), (j^\nu)) = \sum_{i,j=-q}^{N+2q} \lambda_{ij} i^r j^\nu, \tag{6.6}$$

*of the vectors $(j^r)$ and $(j^\nu)$. In terms of the monic polynomial*

$$Q_k(x) = \frac{\begin{vmatrix} (0,0) & (0,1) & \cdots & (0,k-1) & (0,k) \\ (1,0) & (1,1) & \cdots & (1,k-1) & (1,k) \\ \vdots & & & & \\ (k-1,0) & (k-1,1) & \cdots & (k-1,k-1) & (k-1,k) \\ 1 & x & \cdots & x^{k-1} & x^k \end{vmatrix}}{\begin{vmatrix} (0,0) & \cdots & (0,k-1) \\ \vdots & & \\ (k-1,0) & \cdots & (k-1,k-1) \end{vmatrix}} \tag{6.7}$$

*we may express the Legendre splines* (5.12) *as*

$$G_k(x) = \sum_{-q}^{N+q} Q_k(j)\, M_n(x-j). \tag{6.8}$$

*Proof.* It suffices to show that the vector $(Q_k(j))$ is orthogonal to $(j^\nu)$ if $\nu < k$. This we see from (6.7) because $((Q_k(j)), (j^\nu))$ equals the right side of (6.7) if we replace the last row of the first determinant by $(\nu, 0), (\nu, 1),..., (\nu, k)$, and the result evidently vanishes.

Also from the representation (6.8), (6.7) we can derive the property (1.9). However, this approach does not yield Theorem 4 but only the weaker relation $g_{k,N}(x) = X_k(x) + O(N^{-2})$, if $n \geq 2$.

C. de Boor made orally the following interesting remark: If we want to obtain the Legendre splines $G_k(x)$ by applying the Gram–Schmidt process, then the numerical work required will be greatly reduced if we replace the vectors (6.5) that are to be orthogonalized, by the vectors

$$Y_k(j) \qquad (k = 0,..., N+2q), \tag{6.9}$$

where $Y_k(x) = N^k X_k(x/N)$ are the monic Legendre polynomials for the interval $[0, N]$. The reason is that the vectors (6.9) are already nearly orthogonal.

## 7. Complete Spline Interpolation Gives a Different Orthogonal Basis

In Section 5 we have considered the spline interpolant $S_k(x)$, of (5.10), interpolating $x^k$ within $\mathscr{S}_{n-1}^{(k)}[0, N]$. For want of a better name we shall call $S_k(x)$ the *global* interpolant of $x^k$, because it is the restriction to $[0, N]$ of the *cardinal* interpolant of $x^k$. However, we could proceed differently and consider the *complete* spline interpolant $\tilde{S}_k(x)$ of $x^k$. This is the unique element of $\mathscr{S}_{n-1}[0, N]$ that satisfies the conditions

$$\tilde{S}_k(j) = x^k \,|_{x=j} \qquad (j = 0,..., N) \tag{7.1}$$

and the boundary conditions

$$\tilde{S}_k^{(r)}(0) = D^r x^k \,|_{x=0}\,, \tilde{S}_k^{(r)}(N) = D^r x^k \,|_{x=N} \qquad (r = 1,..., q). \tag{7.2}$$

*Having obtained the set*

$$\tilde{S}_k(x) \qquad (k = 0, 1,..., N+2q), \tag{7.3}$$

*we orthogonalize it by the Gram–Schmidt process to obtain the orthogonal basis*

$$\tilde{G}_k(x) \qquad (k = 0, \ldots, N + 2q) \tag{7.4}$$

*for* $\mathscr{S}_{n-1}[0, N]$. Passing to $[0, 1]$ by

$$\tilde{g}_{k,N}(x) = (1/N^k)\, \tilde{G}_k(Nx) \qquad (0 \leqslant x \leqslant 1), \tag{7.5}$$

we easily find that the new orthogonal functions (7.5) also satisfy the Properties 1 and 2 of Section 1 and also Theorem 4 of Section 5.

The following result seems of some interest.

THEOREM 6. *The set* (5.12) *of Legendre splines for* $[0, N]$ *is not always identical with the orthogonal basis* (7.4) *derived by complete spline interpolation.*

*Proof.* We choose the case $n = 4$ of *cubic* spline interpolation and wish to show that

$$G_{5,N}(x) \neq \tilde{G}_{5,N}(x), \tag{7.6}$$

as soon as

$$N \geqslant 5. \tag{7.7}$$

We know that

$$G_{5,N}(x) \text{ is a monic element of } \mathscr{S}_3^{(5)}[0, N]. \tag{7.8}$$

Therefore (7.6) will follow as soon as we show that

$$\tilde{G}_{5,N}(x) \notin \mathscr{S}_3^{(5)}[0, N]. \tag{7.9}$$

Let us assume, on the contrary, that

$$\tilde{G}_{5,N}(x) \in \mathscr{S}_3^{(5)}[0, N] \tag{7.10}$$

and try to get a contradiction.

We know that $\tilde{S}_k(x)$ is the complete spline interpolation of $x^k$ $(k = 0, 1, 2, 3, 4, 5)$ and that $\tilde{G}_{5,N}$ is obtained from these by orthogonalization. It follows that

$$\tilde{G}_{5,N}(x) \tag{7.11}$$

is the complete spline interpolant of a monic $\tilde{P}_5(x) \in \pi_5$. On the other hand, our assumption (7.10) implies that

$$\tilde{G}_{5,N}(x) \tag{7.12}$$

is the restriction to $[0, N]$ of a $S(x) \in \mathscr{S}_3^{(5)}$ and let $S(x)$ interpolate $P_5(x) \in \pi_5$ hence

$$S(j) = P_5(j), \qquad \text{for all integer } j. \tag{7.13}$$

It follows from (7.11)–(7.13) and (7.7) that

$$\bar{P}_5(x) = P_5(x). \tag{7.14}$$

We conclude from (7.11) that $S(x)$ *is the cardinal cubic spline interpolant of a monic quintic polynomial* $P_5(x)$ *with the additional property that*

$$S'(0) = P_5'(0), \qquad S'(N) = P_5'(N). \tag{7.15}$$

Let us show that *even the first relation* (7.15)

$$S'(0) = P_5'(0) \tag{7.16}$$

*is impossible.*

Let $P_5(x) = x^5 + P_4(x)$ and let its cubic spline interpolant be

$$S(x) = S_1(x) + S_2(x),$$

where $S_1$ is the interpolant of $x^5$ and $S_2$ the interpolant of $P_4(x)$. From the remark (4.8) concerning Bernoulli functions we conclude, for $n = 4$, that also $S_2'(x)$ interpolates $P_4'(x)$ at all integers. In particular

$$S_2'(0) = P_4'(0).$$

But then the relation (7.16) would imply that $S_1'(0)$ agrees with $Dx^5 \mid_{x=0} = 0$, hence

$$S_1'(0) = 0. \tag{7.17}$$

*This, however, is not the case.* From Corollary 2 of Section 4 for $P(x) = x^5$ we find that

$$S_1(x) = \sum_{-\infty}^{\infty} \left\{ j^5 - \frac{10}{3} j^3 + \frac{5}{3} j \right\} M_4(x - j).$$

Using the values $M_4'(-1) = \frac{1}{2}$, $M_4'(0) = 0$, $M_4'(1) = -\frac{1}{2}$, while $M_4'(j) = 0$ if $|j| > 1$, we find that

$$S_1'(0) = -\tfrac{2}{3}.$$

This completes our proof.

## 8. An Example Concerning Polynomials Versus Splines

On the subject of least-squares approximation we wish to compare the performance of the two classes of functions

$$\mathcal{S}_{n-1}[0, N] \qquad \text{and} \qquad \pi_{N+2q}, \tag{8.1}$$

having the same dimension $N + 2q + 1$ (as before, $q = [(n-1)/2]$), when applied to one and the same function $f(x)$ defined in $[0, N]$. The most obvious test function would seem to be $f(x) = (x - (N/2))^{n+2q+1}$. This we shall carry out for the case when

$$N = 4, \qquad n = 4, \qquad \text{hence} \qquad q = 1. \tag{8.2}$$

For convenience we replace the class $\mathcal{S}_3[0, 4]$ by the class $\mathcal{S}_3[-2, 2]$ and wish to find the cubic spline $s(x)$, with knots at $-1, 0, 1$, which is the least-squares approximation of $x^7$ in the interval $[-2, 2]$. Writing

$$s(x) = \sum_{-3}^{3} c_j M_4(x - j) \qquad (-2 \leqslant x \leqslant 2), \tag{8.3}$$

we obtain by (2.8) the system

$$\sum_{j=-3}^{3} \lambda_{ij} c_j = I_i \qquad (i = -3, -2, \ldots, 3), \tag{8.4}$$

having the matrix $\Lambda_{4,4}$ explicitly listed in (2.17). For the integrals

$$I_i = \int_{-2}^{2} x^7 M_4(x - i)\, dx,$$

we find by direct evaluation the values $I_0 = 0$ and

$$I_1 = -I_{-1} = \frac{75\,525}{7920}, \quad I_2 = -I_{-2} = \frac{152\,572}{7920}, \quad I_3 = -I_{-3} = \frac{23\,297}{7920}. \tag{8.5}$$

Due to the skew symmetry of the $I_i$ and the central symmetry of the matrix $\Lambda_{4,4}$, also the $c_j$ must be skew-symmetric. Using this fact, the $7 \times 7$ system (8.4) reduces to a $3 \times 3$ system. This reduced system is easily solved exactly and we find the values $c_0 = 0$ and

$$c_1 = -c_{-1} = \frac{616\,462}{573\,573}, \quad c_2 = -c_{-2} = \frac{1\,133\,993}{573\,573}, \quad c_3 = -c_3 = \frac{416\,007\,774}{573\,573}. \tag{8.6}$$

These coefficients and (8.3) give us the spline $s(x)$ of best $L_2$-approximation to $x^7$ in $[-2, 2]$.

For a more convenient comparison with the polynomial case we pass to *the class* $\mathscr{S}_{3,4}[-1, 1]$ *of cubic splines in* $[-1, 1]$ *with knots at the points* $-\frac{1}{2}, 0, +\frac{1}{2}$. Changing scale we find that

$$S(x) = s(2x)/2^7, \qquad (-1 \leqslant x \leqslant 1), \tag{8.7}$$

*is the cubic spline of best approximation to* $x^7$ *in this new class.* For the spline (8.3) we find, by using (8.4), that

$$\int_{-2}^{2} (x^7 - s(x))^2\, dx = \int_{-2}^{2} (x^7)^2\, dx - \sum_{-3}^{3} c_j I_j = 5.445\,481.$$

Changing variables by setting $x = 2t$ and using (8.7) we find that

$$\int_{-1}^{1} (t^7 - S(t))^2\, dt = 2^{-15} \times 5.445\,481 = .000\,166\,1829. \tag{8.8}$$

Let us now find *the polynomial* $P(x) \in \pi_6$ *which is the best* $L_2$*-approximation of* $x^7$ *in the interval* $[-1, 1]$. This $P(x)$ is readily obtained if we observe that $x^7 - P(x)$ is the *monic* Legendre polynomial of degree 7. This implies that $x^7 - P(x) = (16/429)\, P_7(x)$, whence

$$P(x) = x^7 - (16/429)\, P_7(x), \tag{8.9}$$

where $P_7(x)$ is the usual Legendre polynomial. From standard properties we obtain

$$\int_{-1}^{1} (x^7 - P(x))^2\, dx = \left(\frac{16}{429}\right)^2 \int_{-1}^{1} (P_7(x))^2\, dx = \frac{512}{2\,760\,615} = .000\,185\,465\,9. \tag{8.10}$$

We conclude with a table of values of $S(x)$, $P(x)$ and $x^7$. These being odd functions, we may restrict it to $[0, 1]$. A comparison of (8.8) and (8.10) shows that the cubic spline $S(x)$ gives a slightly better approximation than the quintic polynomial $P(x)$. An inspection of the table shows that $P(x)$ oscillates and even becomes negative near $x = 0.5$, while *the spline* $S(x)$ *is strictly increasing in the entire interval* $[-1, 1]$. This seems surprising in view of the fact that the graph of $x^7$ is so very nearly flat in a wide neighborhood of $x = 0$. Because $S(x)$ is an odd function it follows that $S(x)$ is represented by a single cubic polynomial in the interval $(-\frac{1}{2}, \frac{1}{2})$, and only the points $\pm\frac{1}{2}$ are genuine knots.

TABLE 1

| $x$ | $S(x)$ | $P(x)$ | $x^7$ |
|---|---|---|---|
| 0.0 | 0.0000 | 0.0000 | 0.0000 |
| 0.1 | 0.0017 | 0.0074 | 0.0000 |
| 0.2 | 0.0033 | 0.0110 | 0.0000 |
| 0.3 | 0.0050 | 0.0086 | 0.0002 |
| 0.4 | 0.0066 | 0.0031 | 0.0016 |
| 0.5 | 0.0082 | −0.0005 | 0.0078 |
| 0.6 | 0.0172 | 0.0160 | 0.0280 |
| 0.7 | 0.0713 | 0.0767 | 0.0824 |
| 0.8 | 0.2158 | 0.2187 | 0.2097 |
| 0.9 | 0.4956 | 0.4920 | 0.4783 |
| 1.0 | 0.9561 | 0.9627 | 1.0000 |

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*Printed in Belgium*